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Behavior of solutions for a supercritical semilinear heat equation

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This article is based on joint papers [3, 4] with P. Poláčik (University of Minnesota).

Consider the Cauchy problem

$$\text{(E)} \quad \begin{cases} u_t = \Delta u + |u|^{p-1}u, & x \in \mathbf{R}^N,\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbf{R}^N, \end{cases}$$

where $p > 1$. It is known that for the Sobolev exponent

$$p_S = \begin{cases} \dfrac{N+2}{N-2} & \text{if } N > 2, \\ \infty & \text{if } N \le 2, \end{cases}$$

(E) has a one-parameter family of positive radial steady states, i.e., solutions of

$$\Delta\varphi + \varphi^p = 0 \quad \text{on } \mathbf{R}^N,$$

if and only if $p \ge p_S$. We denote the solution by φ_α, $\alpha > 0$, where $\varphi_\alpha(0) = \alpha$. Then φ_α is strictly decreasing in $|x|$ and satisfies $\varphi(|x|) \to 0$ as $|x| \to \infty$. We extend the family by setting

$$\varphi_\alpha = -\varphi_{-\alpha} \ \text{ for } \alpha < 0 \quad \text{and} \quad \varphi_0 \equiv 0.$$

In this article, the following critical value of p is important:

$$p_c = \begin{cases} \dfrac{(N-2)^2 - 4N + 8\sqrt{N-1}}{(N-2)(N-10)} & \text{if } N > 10, \\ \infty & \text{if } N \leq 10. \end{cases}$$

Gui, Ni and Wang [1, 2] have exposed $p = p_c$ as the exponent where a change in stability properties of the positive steady states occurs. While for $p < p_c$ all positive φ_α are unstable in "any reasonable sense", for $p \geq p_c$ they are stable under perturbations in some weighted L^∞ spaces. These stability properties essentially come from the fact that φ_α is strictly increasing in α for each x. Furthermore, for each x one has

$$\lim_{\alpha\to 0} \varphi_\alpha(x) = 0, \qquad \lim_{\alpha\to\infty} \varphi_\alpha(x) = \varphi_\infty(x),$$

where

$$\varphi_\infty(x) = L|x|^{-2/(p-1)} \quad \text{with } L = \left\{\frac{2}{p-1}\left(N - 2 - \frac{2}{p-1}\right)\right\}^{1/(p-1)}.$$

In the present study, we investigate solutions of (E) from a global point of view, focusing exclusively on the case $p \geq p_c$ (thus assuming $N \geq 11$). Building on the results of Gui-Ni-Wang, we first extend their local stability results to global attractivity properties of steady states. Let φ_α be a steady state and consider an initial function u_0 given by

$$u_0 = \varphi_\alpha + v_0.$$

Here v_0 is a (not necessarily small or radial) perturbation that we assume to be continuous. Then for a positive constant

$$\lambda_0 = \lambda_0(N,p) := \frac{N - 2 - \sqrt{(N-2-2m)^2 - 8(N-2-m)}}{2}, \quad m = \frac{2}{p-1},$$

the following theorem holds.

Theorem 1 ([3]) *Let $p \geq p_c$. Assume v_0 satisfies*

$$-\varphi_\infty \leq \varphi_\alpha + v_0 \leq \varphi_\infty$$

and

$$\lim_{|x|\to\infty} |x|^{\lambda_0}|v_0(x)| = 0.$$

Then the solution u of (E) exists globally in time and satisfies

$$\|u(\cdot,t) - \varphi_\alpha\|_{L^\infty(\mathbf{R}^N)} \to 0 \quad \textit{as } t \to \infty.$$

This result can be extended to more general time-dependent (not necessarily positive) solutions that are between $-\varphi_\infty$ and φ_∞.

The next result gives a sharp condition on solutions to decay to 0 as $t \to \infty$.

Theorem 2 ([3]) *Assume $u_0 \in C_0(\mathbf{R}^N)$ satisfies*

$$\begin{aligned}
&-\varphi_\infty(x) \leq u_0(x) \leq +\varphi_\infty(x) \quad \textit{in } R^N,\\
&\lim_{|x|\to\infty} |x|^\lambda \{\, \varphi_\infty(x) - u_0(x) \,\} = \infty,\\
&\lim_{|x|\to\infty} |x|^\lambda \{\, \varphi_\infty(x) + u_0(x) \,\} = \infty.
\end{aligned}$$

Then

$$\|u(\cdot,t,u_0)\|_{L^\infty(\mathbf{R}^N)} \to 0 \quad \textit{as } t \to \infty.$$

By using Theorem 1 and the continuity of solutions with respect to initial data, we can show the existence of global solutions that behaves in a rather complicated way.

Theorem 3 ([4]) *Let $p \geq p_c$. For any (finite or infinite) sequence $\{(\alpha_i, \xi_i, \varepsilon_i)\}$, where $\alpha_i \in \mathbf{R}$, $\xi_i \in \mathbf{R}^N$ and $\varepsilon_i > 0$, there exist initial data u_0 such that the solution of (E) satisfies the following properties:*

(i) $u(x,t)$ *exists globally in time and satisfies* $u \to 0$ *as* $x \to \infty$ *for each* $t > 0$.

(ii) *There exists a sequence of positive numbers* $\{t_i\}$ *such that*

$$\|u(\cdot, t_i) - \varphi_{\alpha_i}(\cdot - \xi_i)\|_{L^\infty(\mathbf{R}^N)} < \varepsilon_i.$$

(iii) *There exists a sequence of positive numbers* $\{\hat{t}_i\}$ *with* $\hat{t}_i \in (t_i, t_{i+1})$ *such that*

$$\|u(\cdot, \hat{t}_i)\|_{L^\infty(\mathbf{R}^N)} < \varepsilon_i.$$

The solutions in the above theorems have at most one bumps at each time. In the next theorem, we show the existence of solutions with multiple bumps.

Theorem 4 ([4]) *Let* $p \geq p_c$. *For any (finite or infinite) sequence* $\{\{\alpha_i^{(j)}\}_{j=1}^{n_i}\}$, *and* $\{\varepsilon_i\}$, *where* n_i *is an arbitrary natural number,* $\alpha_i^{(j)} \in \mathbf{R}$, *and* $\varepsilon_i > 0$, *there exist initial data* u_0 *such that the solution of (E) satisfies the following properties:*

(i) $u(x,t)$ *exists globally in time and satisfies* $u \to 0$ *as* $x \to \infty$ *for each* $t > 0$.

(ii) *There exists a sequence* $\{\{\xi_i^{(j)}\}_{j=1}^{n_i}\} \in \mathbf{R}^N$ *and a sequence of positive numbers* $\{t_i\}$ *such that*

$$\Big\|u(\cdot, t_i) - \sum_{j=1}^{n_i} \varphi_{\alpha_i^{(j)}}(\cdot - \xi_i^{(j)})\Big\|_{L^\infty(\mathbf{R}^N)} < \varepsilon_i.$$

(iii) *There exists a sequence of positive numbers* $\{\hat{t}_i\}$ *with* $\hat{t}_i \in (t_i, t_{i+1})$ *such that*

$$\|u(\cdot, \hat{t}_i)\|_{L^\infty(\mathbf{R}^N)} < \varepsilon_i.$$

References

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[2] C. Gui, W.-M. Ni, and X. Wang. Further study on a nonlinear heat equation. *J. Differential Equations*, 169:588–613, 2001.

[3] P. Poláčik and E. Yanagida On bounded and unbounded global solutions of a supercritical semilinear heat equation, Math. Annal, to appear.

[4] P. Poláčik and E. Yanagida, in preparation.